Abstract— The present work develops an analytical solution based on higher order shear deformation theory for the piezoelectric laminated composite plates. In the Higher Order Shear Deformation Theory, the displacement model is considered in account for non linear variation of in plane and transverse displacements through the plate thickness. The major objective of present study is to determine the vibration characteristics of piezoelectric laminated plates with different lamination schemes, degree of orthotropy and boundary conditions. The derivation of equation of motion of higher order model is obtained by using the principle of virtual work and the solutions are obtained by using Navier’s method. The results are computed and tabulated in non dimensional form for different modulus ratios, a/h ratios and voltages.

Keywords— Piezoelectric laminated composite plates, Higher Order Shear Deformation Theory, principle of virtual work, Navier solutions.

I. INTRODUCTION

The use of piezoelectric materials embedded with composite material are became a new development in recent engineering applications especially in aerospace industry. The use of smart composite laminated plates will give better control over the passive deformation of the structure when they are subjected to external loading. The control of induced vibrations are very much significant in such applications.

Rajan L.Wankhade, Kamal M.B(2016), have presented a paper on Shape control and Vibration Analysis of Piezolaminated Plates Subjected to Electro-Mechanical Loading [1]. Song Xiang, Chun Lu (2015), have proposed a paper on Free vibration of laminated composite plates by the various shear deformation theories [2]. Jafar Rouzegar, Farhad Abad (2015), have investigated an Analysis of Cross-Ply Laminates With Piezoelectric Fiber Reinforced composite Actuators Using Four Variable Refined Plate Theory [3]. P.PhungVan, L.Delorenzis (2014), have submitted a paper on Analysis of laminated composite plates integrated with piezoelectric sensors and actuators using higher order shear deformation theory and geometric finite elements [4]. Shiu-Chuan Her, Chi-Sheng Lin (2013), have presented a paper on Vibration analysis of Composite Laminated Plates Exited by Piezoelectric Actuators [5]. P.Wluka, T.Kubiak (2012), have analyzed the Stability of Cross-Ply Composite plate with Piezoelectric actuators [6]. A.J.M.Ferreira, C.M.C.Roque (2011), are presented a paper on Two higher order Zig-Zag theories for the accurate analysis of bending, vibration and buckling response of laminated plates by radial basis functions collocation and a unified formulation [7]. D.Ngo-Cong, N.Mai-Duy (2010), are further carried out the Free Vibration Analysis of Laminated Composite Plates based on FSDT using One-Dimensional IRBFN Method [8].

The main objective of the present work is to determine the vibration characteristics of piezoelectric laminates plates with different lamination schemes, geometrical parameters, various boundary conditions and degree of orthotropy. A unidirectional laminated composite plate with distributed actuator layers is analyzed under combined electrical and mechanical loading. The obtained results are tabulated in non-dimensional form for different modulus ratios, a/h ratios and voltages.

II. FORMULATION:

2.1 Displacement Field:

A simply supported rectangular cross ply laminated substrate plate is considered for flexural analysis as shown in Fig.1. The length, breadth and thickness of the plate is denoted by a, b and h respectively. The reference axes are assumed as x, y and z are along the plate dimensions a, b and h respectively. A layer of piezoelectric fiber reinforced composite (PFRC) material is embedded to the top surface of the plate, which is acting as the distributed actuator of the plate. The −z− axis is assumed at the middle of the plate i.e. it is located at the distances +h/2 and −h/2 from the top and bottom of the composite cross ply laminate. The thickness of the PFRC actuator is assumed as t_p.
The displacement vectors \( u(x, y, z), v(x, y, z) \) and \( w(x, y, z) \) at any point in the laminate are expanded in the powers of \( z \)-axis. This is to approximately analyze the three dimensional elasticity problem as a two dimensional smart composite laminated problem.

The displacement vectors are expanded in the following form.

**MODEL- 1 (HOSNT):**

\[
\begin{align*}
&u(x, y, z) = u_0(x, y) + z \theta_0(x, y) + z^2 \theta_1(x, y) + \theta_2(x, y) \\
v(x, y, z) = v_0(x, y) + z \theta_0(x, y) + z^2 \theta_1(x, y) + \theta_2(x, y) \\
w(x, y, z) = w_0(x, y) + z \theta_0(x, y) + z^2 \theta_1(x, y) + \theta_2(x, y)
\end{align*}
\]

**MODEL- 2 (HOSNT):**

\[
\begin{align*}
&u(x, y, z) = u_0(x, y) + z \theta_0(x, y) + z^2 \theta_1(x, y) + \theta_2(x, y) \\
v(x, y, z) = v_0(x, y) + z \theta_0(x, y) + z^2 \theta_1(x, y) + \theta_2(x, y) \\
w(x, y, z) = w_0(x, y) + z \theta_0(x, y) + z^2 \theta_1(x, y) + \theta_2(x, y)
\end{align*}
\]

Where the parameters \( u_0, v_0 \) denotes the in plane displacements and \( w_0 \) is the transverse displacement at any point on the mid plane of the substrate. The functions \( \theta x, \theta y \) are the rotations of the normal to the mid plane about \( y \) and \( x \)-axes respectively. The remain parameters \( u_1, v_0, w_0, \theta_1, \theta_2 \) and \( \theta_3 \) are the respective higher order parameters related to the transverse deformation modes.

In the present formulation in conformity with the usual plate assumptions, by substitution of displacement relations from the equation (1) in to the strain displacement equations of the classical laminate theory of elasticity, the following relations are obtained:

\[
\begin{align*}
\left[ E_{xx} E_{xy} E_{xz} E_{yy} E_{yz} \right]^T = \left[ \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right]^T \quad (2)
\end{align*}
\]

2.2. Lamina coupled constitutive equations:

The linear constitutive relations for a single piezoelectric layer couples the elastic and electric fields as given below,

\[
\begin{align*}
\sigma & = [D] \epsilon + [Q] \phi, \\
[D] & = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}, \\
\phi & = \begin{bmatrix} \xi_x \\ \xi_y \\ \xi_z \end{bmatrix}.
\end{align*}
\]

(3)

The electric field intensity vector \( E \) related to electrostatic potential \( \xi(x, y, z) \) in the \( L_b \) layer is given by:

\[
E_x = \frac{\partial \xi(x,y,z)}{\partial x} ; \quad E_y = \frac{\partial \xi(x,y,z)}{\partial y} ; \quad E_z = \frac{\partial \xi(x,y,z)}{\partial z}
\]

Where \( \sigma, Q, \epsilon, E, D \) and \( \phi \) are the stress vector, elastic constant matrix, strain vector, piezoelectric constant matrix, electric field intensity vector, electric displacement vector and dielectric constant matrix respectively.

2.3. Governing equations of motion:

The governing equations of motion of higher order theory will be derived using principle of virtual work.

\[
\int_0^T \delta(u + v - k) dt = 0
\]

The equations of motion in terms of virtual displacements can be expressed are obtained as

\[
\begin{align*}
\delta u_0 & = -\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_1 u_0 + I_2 \theta_0 + I_3 u_1 + I_4 \theta_1 \\
\delta v_0 & = -\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_1 v_0 + I_2 \theta_0 + I_3 v_1 + I_4 \theta_1 \\
\delta w_0 & = -\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_1 w_0 + I_2 \theta_0 + I_3 w_1 + I_4 \theta_1
\end{align*}
\]

2.4. Admissible boundary conditions for the Navier's solutions of the displacement model:

To discuss the Navier and other solutions, consider those which satisfy the boundary conditions of the problem. The Navier solutions can be developed for the rectangular laminates with two sets of simply supported boundary conditions. The following are the mechanical and electrical in plane boundary conditions for the simply supported plate.

**SS-1 Boundary condition:** At edges \( x = 0 \) and \( x = a; \)

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\[ v_0 = 0, \quad w_0 = 0, \quad \theta_y = 0, \quad \theta_z = 0, \quad M_x = 0, \quad N_x = 0, \]
\[ v^*_0 = 0, \quad w^*_0 = 0, \quad \theta^*_y = 0, \quad M^*_x = 0, \quad N^*_x = 0, \]
\[ \xi = 0 \]

**SS-2 Boundary condition:** At edges \( y = 0 \) and \( y = b \):
\[ u_0 = 0, \quad w_0 = 0, \quad \theta_x = 0, \quad \theta_z = 0, \quad M_y = 0, \quad N_y = 0, \]
\[ v^*_0 = 0, \quad w^*_0 = 0, \quad \theta^*_x = 0, \quad M^*_y = 0, \quad N^*_y = 0, \]
\[ \xi = 0 \]

By considering the above boundary conditions and using the Navier’s method, the mechanical load and mid plane displacements are expanded as follows:
\[ U_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{0mn} \cos \frac{\max \{ m \alpha \}}{a} \sin \frac{\max \{ n \beta \}}{b}, \]
\[ U^*_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U^*_{0mn} \cos \frac{\max \{ m \alpha \}}{a} \sin \frac{\max \{ n \beta \}}{b}, \]
\[ V_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{0mn} \sin \frac{\max \{ m \alpha \}}{a} \cos \frac{\max \{ n \beta \}}{b}, \]
\[ V^*_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V^*_{0mn} \sin \frac{\max \{ m \alpha \}}{a} \cos \frac{\max \{ n \beta \}}{b}, \]
\[ W_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{0mn} \sin \frac{\max \{ m \alpha \}}{a} \sin \frac{\max \{ n \beta \}}{b}, \]
\[ W^*_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W^*_{0mn} \sin \frac{\max \{ m \alpha \}}{a} \sin \frac{\max \{ n \beta \}}{b}, \]
\[ \theta_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{xmn} \cos \frac{\max \{ m \alpha \}}{a} \sin \frac{\max \{ n \beta \}}{b}, \]
\[ \theta^*_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta^*_{xmn} \cos \frac{\max \{ m \alpha \}}{a} \sin \frac{\max \{ n \beta \}}{b}, \]
\[ \theta_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{ymn} \sin \frac{\max \{ m \alpha \}}{a} \cos \frac{\max \{ n \beta \}}{b}, \]
\[ \theta^*_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta^*_{ymn} \sin \frac{\max \{ m \alpha \}}{a} \cos \frac{\max \{ n \beta \}}{b}, \]
\[ \theta_z = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{zmn} \sin \frac{\max \{ m \alpha \}}{a} \sin \frac{\max \{ n \beta \}}{b}, \]
\[ \theta^*_z = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta^*_{zmn} \sin \frac{\max \{ m \alpha \}}{a} \sin \frac{\max \{ n \beta \}}{b}, \]
\[ q_z = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{zmn} \sin \frac{\max \{ m \alpha \}}{a} \sin \frac{\max \{ n \beta \}}{b}, \]

And electrostatic potential is given as
\[ \xi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \xi_{mn} (z) \sin \frac{\max \{ m \alpha \}}{a} \sin \frac{\max \{ n \beta \}}{b}. \]

The above expansions are substituted into the governing equation of motions and which gives the required system equations.

### 2.5. Free vibration analysis of laminated plates using higher order displacements:

For free vibration analysis of laminated composite plate problems, the equations of motion can be reduced to eigen value problem as;
\[
[S] - \omega^2[M]\{\Delta\} = \{0\}
\]

Where the real positive roots of the equation gives the square of the natural frequency. The smallest magnitude of eigen value of the equation is called the fundamental natural frequency.

### III. RESULTS AND DISCUSSIONS

In the present work a cross ply and angle ply laminated composite plates are considered individually for vibration analysis and the plates to be simply supported all of its sides. The plates are to be polymer based layers of Glass/Epoxy. At the top of the elastic substrate a Piezoelectric Fiber Reinforced Composite material (PFRC) is layered in a distributed form. The following material properties are used to find the numerical results of the developed higher order theory to compare with the results available in the literature to ascertain its validity. The computer programs for higher order shear deformation theories with 9 and 12 degrees of freedom are developed individually for anti symmetric cross ply and angle ply laminates of model 1 and model 2. The numerical results are carried out for the free un damped transverse vibration analysis of laminated composite plates. The effect of side to thickness ratio, number of layer, shear deformation and coupling on fundamental natural frequencies are investigated.

**Glass/Epoxy:**
\[ E_2 = E_3 = 10^6 \text{ Gpa} \quad ; \quad E_1/E_2 = 40 \quad ; \quad G_{12} = 0.6 \quad E_2; \quad G_{23} = 0.5 \quad E_1; \quad G_{13} = 0.5 \; \]
\[ v_{12} = v_{23} = v_{13} = 0.25; \]

**PFRC material :**
\[ E_f = E_3 = 10^6 \text{ Gpa} \quad ; \quad E_1/E_2 = 25 \quad ; \quad G_{12} = 0.5 \quad E_2; \quad G_{23} = 0.5 \; \]
\[ v_{12} = v_{23} = v_{13} = 0.25; \]

The following relation is used for presentation of non dimensional fundamental frequencies in this paper.
\[
\bar{\omega} = \frac{\omega b^2}{h} \sqrt{\rho / E_2}
\]

Fig. 2 shows the variation of non dimensional normal frequency for various modulus ratio and for various voltages with and without smart materials is observed. From the figure it is found that the effect of coupling is more significant for all modulus ratios in model 1 and model 2 and it is observed that the piezo effect is decreased for modulus ratio for zero voltage. From Fig.3 it is observed that for the non-piezo
materials (without piezo) the maximum frequencies are occurring at E1/E2= 40 at 100 V and are drastically decreased at E1/E2= 10 and also observed that the piezo effect is very close to that of without piezo. Fig.4 shows that the variation of the non-dimensionalized fundamental natural frequencies of the anti symmetric angle ply laminates for various modulus ratios with and without piezo materials for zero voltage and 100 V. From fig.4, it is found that the frequencies decreased at 100 V compared to zero voltage and also the piezo effect in both cases are slightly decreased with respect to without piezo effect.

**CONCLUSIONS**

The results obtained from the present study conclude that the higher order shear deformation theory predicts are close to that of exact solutions for the smart unidirectional plates mainly for frequency. It is observed that from results, the natural frequency decreases the vibrations when the piezo actuator is presented. when the applied voltage is specified, the natural frequency decreases with increase in voltage. The thickness of the piezoelectric layer is small compared with the thickness of the elastic layer, hence the variation is not much affected when compared with elastic solutions of non-piezoelectric plate. By the comparison the present results are effective and more reliable than the exact solutions results available in literature.

**REFERENCES**


