STABILITY OF DELAYS NEURAL NETWORKS

GRIENGGRAI RAJCHAKIT

Department of Mathematics, Faculty of Science, Maejo University, Chiang Mai 50290, Thailand
Email: kreangkri@mju.ac.th

Abstract— In this paper, we derive a sufficient condition for asymptotic stability of the zero solution of delay-difference system with multiple delays of Hopfield neural networks in terms of certain matrix inequalities by using a discrete version of the Lyapunov second method.

Keywords— Asymptotic stability; Delay-difference system with multiple delays; Hopfield neural networks; Lyapunov function; Matrix inequalities.

I. INTRODUCTION

In this paper, we consider delay-difference system with multiple delays of Hopfield neural networks of the form

$$u(k+1) = -Cu(k) + \sum_{i=1}^{m} B_i S(u(k-h_i)) + f\,, \tag{1}$$

where $u(k) \in \Omega \subseteq \mathbf{R}^n$ is the neuron state vector, $0 \le h_1 \le \ldots \le h_m$, $C = diag\{a_1, \ldots, a_n\}$, $c_i \ge 0$, $i = 1, 2, ..., n$ is the constant relaxation matrix, B_i, $i = 1, 2, ..., m$ are $n \times n$ constant weight matrices, $f = (f_1, \ldots, f_n) \in \mathbf{R}^n$ is the constant external input vector and $S(z) = [s_1(z_1), \ldots, s_n(z_n)]^T$ with $s_i \in C^1[\mathbf{R}, (-1,1)]$ where s_i is the neuron activations and monotonically increasing for each $i = 1, 2, ..., n$.

II. PRELIMINARIES

Fact 2.1 For any positive scalar ε and vectors x and y, the following inequality holds:

$$x^T y + y^T x \le \varepsilon x^T x + \varepsilon^{-1} y^T y\,.$$

Let us denote $V_\delta = \{x \in \mathbf{R}^n : \ \|x\| < \delta\}$.

Lemma 2.1 The zero solution of difference system is asymptotic stability if there exists a positive definite function $V(x): \mathbf{R}^n \to \mathbf{R}^+$ such that

$$\exists \beta > 0 : \Delta V(x(k)) =$$

$$V(x(k+1)) - V(x(k)) \le -\beta \|x(k)\|^2\,,$$

along the solution of the system. In case the above condition holds for all $x(k) \in V_\delta$, we say that the zero solution is locally asymptotically stable.

Lemma 2.2 For any constant symmetric matrix $M \in \mathbf{R}^{n \times n}$, $M = M^T > 0$, scalar $s \in \mathbf{Z}^+ / \{0\}$, vector function $W : [0, s] \to \mathbf{R}^n$, we have

$$s\sum_{i=0}^{s-1} (w^T(i) M w(i)) \ge \left(\sum_{i=0}^{s-1} w(i) \right)^T M \left(\sum_{i=0}^{s-1} w(i) \right).$$

We present the following technical fact and lemmas, which will be used in the proof of our main result.

III. RESULTS AND DISCUSSION

In this section, we consider the sufficient condition for asymptotic stability of the zero solution u^* of (1) in terms of certain matrix inequalities. Without loss of generality, we can assume that $u^* = 0, S(0) = 0$ and f =0 (for otherwise, we let $x = u - u^*$ and define $S(x) = S(x + u^*) - S(u^*)$.

The new form of (1) is now given by

$$u(k+1) = -Cu(k) + \sum_{i=1}^{m} B_i S(u(k-h_i))\,. \tag{2}$$

Throughout this paper we assume the neuron activations $s_i(x_i)$, $i = 1, 2, \ldots, n$ is bounded and monotonically nondecreasing on $\mathbf{R}$, and $s_i(x_i)$ is Lipschitz continuous, that is, there exist constant $l_i > 0$, $i = 1, 2, \ldots, n$ such that

$$|s_i(v_1) - s_i(v_2)| \le l_i |v_1 - v_2|, \qquad \forall v_1, v_2 \in \mathbf{R}\,. \tag{3}$$

By condition (3), $s_i(x_i)$ satisfy

$$|s_i(x_i)| \le l_i |x_i|, \qquad i = 1, 2, ..., n\,. \tag{4}$$

Theorem 3.1 The zero solution of the delay-difference system (2) is asymptotically stable if there exist the symmetric positive definite matrices $P(k)$, $G_i(k)$, $W_i(k)$, $i = 1, 2, ..., m$, and

$L = diag[l_1,\ldots,l_n] > 0$ satisfying the following matrix inequalities:

$$\psi=\begin{bmatrix} (0,0) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & (1,1) & (1,2) & \cdots & (1,m) & 0 & 0 & 0 & \cdots & 0 \\ 0 & (2,1) & (2,2) & \cdots & (2,m) & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (m,1) & (m,2) & \cdots & (m,m) & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & (m+1,m+1) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (m+2,m+2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & (2m,2m) \end{bmatrix} < 0, \tag{5}$$

where

$$(0,0) = C^T P(k)C - P(k) + \sum_{i=1}^{m}\left(h_i G_i(k) + W_i(k)\right)$$

$$(1,1) = LB_1^T P(k)B_1 L + \varepsilon_1^{-1} LL - W_1(k),$$

$$(1,2) = LB_1^T P(k)B_2 L + \varepsilon_1^{-1} LL,$$

$$(1,m) = LB_1^T P(k)B_m L + \varepsilon_1^{-1} LL,$$

$$(2,1) = LB_2^T P(k)B_1 L + \varepsilon_1^{-1} LL,$$

$$(2,2) = LB_2^T P(k)B_2 L + \varepsilon_1^{-1} LL - W_2(k),$$

$$(2,m) = LB_2^T P(k)B_m L + \varepsilon_1^{-1} LL,$$

$$(m,1) = LB_m^T P(k)B_1 L + \varepsilon_1^{-1} LL,$$

$$(m,2) = LB_m^T P(k)B_2 L + \varepsilon_1^{-1} LL,$$

$$(m,m) = LB_m^T P(k)B_m L + \varepsilon_1^{-1} LL - W_m(k),$$

$$(m+1,m+1) = -h_1 G_1(k),$$

$$(m+2,m+2) = -h_2 G_2(k),$$

$$(2m,2m) = -h_m G_m(k).$$

Proof Consider the Lyapunov function

$V(y(k)) = V_1(y(k)) + V_2(y(k)) + V_3(y(k))$,

where

$$V_1(y(k)) = x^T(k)P(k)x(k),$$

$$V_2(y(k)) = \sum_{i=1}^{m}\sum_{j=k-h_i+1}^{k}(h-k+i)x^T(j)G_i(k)x(j),$$

$$V_3(y(k)) = \sum_{i=1}^{m}\sum_{j=k-h_i+1}^{k}x^T(j)W_i(k)x(j),$$

$P(k), G_i(k), W_i(k)$, $i = 1,2,...,m$ being symmetric positive definite solutions of (5) and $y(k) = \left[x(k), x(k-h_1),\ldots,x(k-h_m)\right]$.

Then difference of $V(y(k))$ along trajectory of solution of (2) is given by

$$\Delta V(y(k)) = \Delta V_1(y(k)) + \Delta V_2(y(k)) + \Delta V_3(y(k))$$

, where

$$\Delta V_1(y(k)) = V_1(x(k+1)) - V_1(x(k))$$

$$= [-Cx(k) + \sum_{i=1}^{m} B_i S(x(k-h_i))]^T P(k)$$

$$\times[-Cx(k) + \sum_{i=1}^{m} B_i S(x(k-h_i))]$$

$$-x^T(k)P(k)x(k)$$

$$= x^T(k)[CP(k)C - P(k)]x(k)$$

$$-\sum_{i=1}^{m}x^T(k)CP(k)B_i S(x(k-h_i)) - \sum_{i=1}^{m}S^T(x(k-h_i))B_i^T P(k)Cx(k)$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{m}S^T(x(k-h_i))C_i^T PC_j S(x(k-h_j)),$$

$$\Delta V_2(y(k)) = \Delta\left(\sum_{i=1}^{m}\sum_{j=k-h_i+1}^{k}(h_i - k + j)x^T(j)G_i(k)x(j)\right)$$

$$= \sum_{i=1}^{m}h_i x^T(k)G_i(k)x(k) - \sum_{i=1}^{m}\sum_{j=k-h_i+1}^{k}x^T(j)G_i(k)x(j),$$

$$\Delta V_3(y(k)) = \Delta\left(\sum_{i=1}^{m}\sum_{j=k-h_i+1}^{k}x^T(j)W_i(k)x(j)\right)$$

$$= \sum_{i=1}^{m}x^T(k)W_i(k)x(k) - \sum_{i=1}^{m}x^T(k-h_i)W_i(k)x(k-h_i) \tag{6}$$

where (4) and **Fact 2.1** are utilized in (6), respectively.

Note that

$$-\sum_{i=1}^{m}x^T(k)CP(k)B_i S(x(k-h_i)) - \sum_{i=1}^{m}S^T(x(k-h_i))B_i^T P(k)Cx(k)$$

$$\le \varepsilon_1\sum_{i=1}^{m}\sum_{j=1}^{m}x^T(k)CP(k)B_i B_j^T P(k)Cx(k)$$

$$+\varepsilon_1^{-1}\sum_{i=1}^{m}\sum_{j=1}^{m}S^T(x(k-h_i))S(x(k-h_j)),$$

$$\sum_{i=1}^{m}\sum_{j=1}^{m}S^T(x(k-h_i))B_i^T P(k)B_j S(x(k-h_j)) \le$$

$$\sum_{i=1}^{m}\sum_{j=1}^{m}x^T(k-h_i)LB_i^T P(k)B_j Lx(k-h_j)$$

,

$$\varepsilon^{-1}S^T(x(k))S(x(k)) \le \varepsilon^{-1}x^T(k)LLx(k),$$

hence

$$\Delta V_1 \le x^T(k)[CP(k)C - P(k)]x(k)$$
$$+\varepsilon_1 \sum_{i=1}^{m}\sum_{j=1}^{m} x^T(k)CP(k)B_iB_j{}^T P(k)Cx(k)$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{m} x^T(k-h_i)LB_i{}^T P(k)B_jLx(k-h_j)$$
$$+\varepsilon_1{}^{-1}\sum_{i=1}^{m}\sum_{j=1}^{m} x^T(k-h_i)LLx(k-h_j)$$
$$+\varepsilon^{-1}x^T(k)LLx(k),$$

Then we have

$$\Delta V \le x^T(k)[CP(k)C - P(k) + \sum_{i=1}^{m}(h_iG_i(k) + W_i(k))$$
$$+\varepsilon_1 \sum_{i=1}^{m}\sum_{j=1}^{m} CP(k)B_iB_j{}^T P(k)C]x(k)$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{m} x^T(k-h_i)[LB_i{}^T P(k)B_jL + \varepsilon_1{}^{-1}LL]x(k-h_j)$$
$$-\sum_{i=1}^{m} x^T(k-h_i)W_i(k)x(k-h_i)$$
$$-\sum_{i=1}^{m}\sum_{j=k-h_i+1}^{k} x^T(j)G_i(k)x(j).$$

Using **Lemma 2.2**, we obtain

$$x^T(i)G_i(k)x(i) \ge^T \sum_{i=k-h+1}^{k}\left(\frac{1}{h}\sum_{i=k-h+1}^{k} x(i)\right) \quad (hG_i(k))\left(\frac{1}{h}\sum_{i=k-h+1}^{k} x(i)\right).$$

From the above inequality it follows that:

$$\Delta V \le x^T(k)[CP(k)C - P + \sum_{i=1}^{m} h_iG_i(k) + W_i(k)$$
$$+\varepsilon_1 \sum_{i=1}^{m}\sum_{j=1}^{m} CP(k)B_iB_j{}^T P(k)C + \varepsilon^{-1}LL]x(k)$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{m} x^T(k-h_i)[LB_i{}^T P(k)B_jL + \varepsilon_1{}^{-1}LL]x(k-h_j)$$
$$-\sum_{i=1}^{m} x^T(k-h_i)W_i(k)x(k-h_i)$$

$$-\sum_{i=1}^{m}\left(\frac{1}{h_i}\sum_{j=k-h_i+1}^{k} x(j)\right)^T (h_iG_i(k))\left(\frac{1}{h_i}\sum_{j=k-h_i+1}^{k} x(j)\right)$$

$$= -\begin{pmatrix} x^T(k), x^T(k-h_1), x^T(k-h_2), \ldots, x^T(k-h_m), (\frac{1}{h_1}\sum_{i=k-h_1+1}^{k} x(j))^T \\ , (\frac{1}{h_2}\sum_{i=k-h_2+1}^{k} x(j))^T, \ldots, (\frac{1}{h_m}\sum_{i=k-h_m+1}^{k} x(j)^T \end{pmatrix}$$

$$\times \begin{bmatrix} (0,0) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & (1,1) & (1,2) & \cdots & (1,m) & 0 & 0 & 0 & \cdots & 0 \\ 0 & (2,1) & (2,2) & \cdots & (2,m) & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (m,1) & (m,2) & \cdots & (m,m) & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & (m+1,m+1) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (m+2,m+2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & (2m,2m) \end{bmatrix}$$

$$\times \begin{pmatrix} x(k) \\ x(k-h_1) \\ x(k-h_2) \\ \vdots \\ x(k-h_m) \\ \left(\frac{1}{h_1}\sum_{j=k-h_1}^{k-1} x(j)\right) \\ \left(\frac{1}{h_2}\sum_{j=k-h_2}^{k-1} x(j)\right) \\ \vdots \\ \left(\frac{1}{h_m}\sum_{j=k-h_m}^{k-1} x(j)\right) \end{pmatrix}.$$

$$= y^T(k)\psi y(k),$$

where

$$(0,0) = CP(k)C - P(k) + \sum_{i=1}^{m} h_iG_i(k) + W_i(k)$$
$$+\varepsilon_1 \sum_{i=1}^{m}\sum_{j=1}^{m} CP(k)B_iB_j{}^T P(k)C,$$
$$(1,1) = LB_1{}^T P(k)B_1L + \varepsilon_1{}^{-1}LL - W_1(k),$$
$$(1,2) = LB_1{}^T P(k)B_2L + \varepsilon_1{}^{-1}LL,$$
$$(1,m) = LB_1{}^T P(k)B_mL + \varepsilon_1{}^{-1}LL,$$
$$(2,1) = LB_2{}^T P(k)B_1L + \varepsilon_1{}^{-1}LL,$$
$$(2,2) = LB_2{}^T P(k)B_2L + \varepsilon_1{}^{-1}LL - W_2(k),$$
$$(2,m) = LB_2{}^T P(k)B_mL + \varepsilon_1{}^{-1}LL,$$
$$(m,1) = LB_m{}^T P(k)B_1L + \varepsilon_1{}^{-1}LL,$$
$$(m,2) = LB_m{}^T P(k)B_2L + \varepsilon_1{}^{-1}LL,$$
$$(m,m) = LB_m{}^T P(k)B_mL + \varepsilon_1{}^{-1}LL - W_m(k)$$

$$(m+1,m+1) = -h_1 G_1(k),$$
$$(m+2,m+2) = -h_2 G_2(k),$$
$$(2m,2m) = -h_m G_m(k),$$

$$y(k) = \begin{pmatrix} x(k) \\ x(k-h_1) \\ x(k-h_2) \\ \vdots \\ x(k-h_m) \\ \left(\frac{1}{h_1}\sum_{j=k-h_1+1}^{k} x(j)\right) \\ \left(\frac{1}{h_2}\sum_{j=k-h_2+1}^{k} x(j)\right) \\ \vdots \\ \left(\frac{1}{h_m}\sum_{j=k-h_m+1}^{k} x(j)\right) \end{pmatrix}.$$

By the condition (5), ΔV is negative definite, namely there is a number $\beta > 0$ such that $\Delta V(y(k)) \le -\beta \|y(k)\|^2$, and hence, the asymptotic stability of the system immediately follows from **Lemma 2.1**. This completes the proof.

Remark 3.1 Theorem 3.1 gives a sufficient condition for the asymptotic stability of delay-difference system (2) via matrix inequalities. These conditions are described in terms of certain diagonal matrix inequalities, which can be realized by using the linear matrix inequality algorithm proposed in [4]. But Hu and Wang [9] these conditions are described in terms of certain symmetric matrix inequalities, which can be realized by using the Schur complement lemma and linear matrix inequality algorithm proposed in [4].

CONCLUSIONS

In this paper, based on a discrete analog of the Lyapunov second method, we have established a sufficient condition for the asymptotic stability of delay-difference system with multiple delays of Hopfield neural networks in terms of certain matrix inequalities.

REFERENCES

[1] Gu K (2000), An Integral Inequality in the Stability Problem of Time-Delay Systems, *In: Proc39th IEEE CDC Conf.*, Sydney, Australia, pp. 2805-2810.

[2] Liu XZ, Liu Y, Teo K L (2003), Stability Analysis of Impulsive Control Systems, *Compute Math Model*, Vol. 37, pp. 1357-1370.

[3] Ratchagit K (2007), "Asymptotic Stability of Delay Difference System of the Hopfield Neural Networks via Matrix Inequalities and Application", International Journal of Neural Systems, Vol. 17, No. 5, pp. 425-430.

[4] Ratchagit K (2007), "Asymptotic Stability of Delay Difference System of Cellular Neural Networks with Multiple Delays via Matrix Inequalities", KMITL Science Journal, Vol. 7, No. 2, pp. 18-27.

[5] Ratchagit K (2007), "Asymptotic Stability of the Delay Difference System of the Cellular Neural Networks via Matrix Inequalities and Application", Advances in Computer Science and Engineering, Vol. 2, No. 1, pp. 51-59.

[6] Ratchagit K (2007), "Asymptotic Stability of the Delay Difference System of the Cellular Neural Networks via Matrix Inequalities and Application", Advances in Differential Equations and Control Processes, Vol. 2, No. 2, pp. 171-177.

[7] Ratchagit K (2008), "Asymptotic Stability of a Delay-Difference Control System of Hopfield Neural Networks via Matrix Inequalities and Applications", Walailak J Sci & Tech, Vol. 5, No. 2, pp. 191-202.

[8] Ratchagit K (2009), "Asymptotic Stability of Delay-Difference System of Cellular Neural Networks with Time-Varying Delay via Matrix Inequalities", International Mathematical Forum, Vol. 4, No. 15, pp. 747-755.

[9] Ratchagit K (2009), "Asymptotic Stability of Time Varying Delay-Difference System of Cellular Neural Networks with Time-Varying Delay via Matrix Inequalities and Applications", Int. J. of Appl. Math and Mech, Vol. 5, No. 3, pp. 11-21.

[10] Ratchagit K (2009), "Asymptotic Stability of Delay-Difference Control System of Hopfield Neural Networks via Matrix Inequalities", International Mathematical Forum, Vol. 4, No. 28, pp. 1405-1411.

[11] Ratchagit K (2009), "Asymptotic Stability of Time Varying Delay-Difference System via Matrix Inequalities", International Mathematical Forum, Vol. 4, No. 29, pp. 1443-1447.

[12] Ratchagit K (2009), "Asymptotic Stability of Nonlinear Delay-Difference Control System with Time-Varying Delay via Matrix Inequalities", Applied Mathematical Sciences, Vol. 3, No. 30, pp. 1473-1478.

[13] Ratchagit K (2009), "Asymptotic Stability of Delay-Difference Control System with Time-Varying Delay via Matrix Inequalities", Int. J. Contemp. Math. Sciences, Vol. 4, No. 20, pp. 999-1004.

★★★